Self-consistent distribution of a high brightness beam in a continuous focusing channel and application to halo-free beam transport

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The self-consistent particle distribution of a high brightness beam in a uniform channel with arbitrary focusing potential is derived. It is shown that the self-potential of a space-charge dominated beam always tends to the same distribution as an external focusing potential with opposite sign regardless of the applied focusing field. Subsequent approximation formulas to the space charge potential of the beam have been derived, which demonstrates the effect of shielding of the external field. The developed approach is checked via known solution as a Gaussian beam distribution matched with a nonlinear focusing channel. The performed study provides a theoretical basis for choosing parameters of the space charge dominated beam transport with suppressed emittance growth. Numerical results demonstrating prevention of halo formation for a bright, nonuniform beam, with a phase space density value of 1.5 A/(π cm mrad) are given.
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I. INTRODUCTION

Emittance conservation of a high brightness particle beam is an important issue for existing and future high intensity accelerator projects. If the beam is matched with the uniform focusing channel, its distribution function as well as beam emittance and beam brightness are conserved. Finding matched conditions for the beam requires solutions of the self-consistent problem for the beam distribution function in phase space. Self-consistent particle distribution creates a potential in which particle motion maintains this distribution.

A nonuniform space charge dominated beam is mismatched with a linear focusing channel, which results in beam emittance growth and halo formation. Recently it was found that nonuniform beam distribution is conserved in a highly nonlinear focusing field [1]. A nonlinear field distribution can be created in a multipole alternating gradient channel [2]. The effective potential of such a structure is a complicated function of radius and azimuth angle. Finding matching conditions for a beam in such a structure is required to provide beam transport without emittance growth and halo formation. In this paper, the general approach to determine a matched beam distribution in a continuous channel with an arbitrary applied focusing potential is developed. Results of the study are applied to a practical solution of the important problem of intense beam transport without halo formation.

II. SINGLE-PARTICLE HAMILTONIAN

Let us consider a high brightness beam of particles with charge q, rest mass m, and beam current I, propagating in a z-uniform focusing channel with longitudinal velocity β. The single-particle Hamiltonian in a focusing channel is given by

$$K = c\sqrt{m^2c^2 + (P_x - qA_x)^2 + (P_y - qA_y)^2 + (P_z - qA_z)^2} + qU_{\text{ext}} + qU_b, \tag{2.1}$$

where c is the velocity of light, $\vec{P} = (P_x, P_y, P_z)$ is a canonical momentum of particles; $\vec{A} = (A_x, A_y, A_z)$ is a vector potential, $U_{\text{ext}} = U_{\text{ext}}(x,y)$ is a scalar potential of the focusing field, and $U_b = U_b(x,y)$ is a space charge potential of the beam. In a moving coordinate system where particles are static, the vector potential of the beam equals zero, $\vec{A} = 0$. According to the Lorentz transformation, components of the vector potential are converted into the laboratory system of coordinates as follows: $A_x = A_y = 0$, $A_z = \beta U_b / c$ [3]. Transverse components of mechanical momentum $\vec{p} = \vec{P} - q\vec{A}$ are equal to that of canonical momentum $p_x = P_x$, $p_y = P_y$. To make a simplification of the Hamiltonian (2.1), let us take into account that kinetic energy of the beam is much larger than the self-potential-energy of the beam. Consider, for simplicity, a uniformly populated beam with space charge potential

$$U_b = -\frac{I}{4\pi\epsilon_0\beta c}\left(\frac{r}{R}\right)^2 = -\frac{mc^2}{q}\frac{I}{I_c\beta}\left(\frac{r}{R}\right)^2 = -\frac{mc^2}{q}\left.\frac{I}{I_c\beta}\right|_{r=R}, \tag{2.2}$$

where R is a beam radius and $I_c = 4\pi\epsilon_0 mc^3/q = 3.13\times10^7 (A/Z)$ A is the characteristic value of beam current. Substitution of Eq. (2.2) into the expression for the longitudinal component of the canonical momentum gives

$$P_z = p_z + qA_z = mc\beta\gamma + q\beta\frac{U_b}{c} = mc\beta\gamma\left(1 - \frac{I}{I_c\beta\gamma}\right). \tag{2.3}$$

We consider beam transport with the beam current a value much lower than the Alfvén current $I \ll \beta\gamma I_c$. Therefore, in Eq. (2.3) $P_z \gg qA_z$, and $(P_z - qA_z)^2 \approx P_z^2 - 2P_z qA_z$. After expanding small terms $\sqrt{1+x} \approx 1 + x/2$ in the Hamiltonian function and using the new Hamiltonian $H=K-mc^2\gamma$, the general form of the single-particle Hamiltonian in a uniform focusing channel is

$$H=\frac{p_x^2+p_y^2}{2m\gamma}+qU_{\text{ext}}+q\,\frac{U_b}{\gamma^2}. \tag{2.4}$$

In Ref. [3], the Hamiltonian (2.4) was used to treat self-consistent beam dynamics in the linear focusing field $U_{\text{ext}}=G\,r^2/2$, where G is a focusing gradient. Below the approach is generalized for the case of an arbitrary applied potential for the focusing field U_{ext}.

III. PARTICLE DISTRIBUTION FUNCTION

The general approach to find a self-consistent distribution function for a time-independent process is to represent it as a function of Hamiltonian $f=f(H)$ [3]. Substitution of the distribution function into Poisson's equation provides a nonlinear equation for unknown space charge potential of the beam U_b, which appears in both the left and right sides of the equation:

$$\Delta U_b=-\frac{q}{\epsilon_0}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f\left(\frac{p_x^2+p_y^2}{2m\gamma}+qU_{\text{ext}}+q\,\frac{U_b}{\gamma^2}\right)dp_x\,dp_y. \tag{3.1}$$

After solving Eq. (3.1) for the charge potential of the beam U_b, one can find the self-consistent particle distribution, which will be maintained in the focusing channel. A convenient way is to use an exponential function:

$$f=f_0\exp\left(-\frac{H}{H_0}\right)=f_0\exp\left(-\frac{p_x^2+p_y^2}{2m\gamma H_0}-q\,\frac{U_{\text{ext}}+\gamma^{-2}U_b}{H_0}\right). \tag{3.2}$$

The distribution function (3.2) contains two unknown constants f_0 and H_0, which can be expressed through the beam parameters. Let us rewrite distribution function (3.2) as follows:

$$f=f_0\exp\left(-2\,\frac{p_x^2}{p_0^2}-2\,\frac{p_y^2}{p_0^2}-q\,\frac{U_{\text{ext}}+\gamma^{-2}U_b}{H_0}\right), \tag{3.3}$$

where $p_0=2\sqrt{\langle p_x^2\rangle}=2\sqrt{\langle p_y^2\rangle}$ is the double rms (root-mean-square) beam size in phase space. Beam radius $R=2\sqrt{\langle x^2\rangle}$ is the double value of the rms beam size in configuration space. The rms value of beam emittance ϵ is a product of beam radius R and p_0:

$$\epsilon=\frac{4}{mc}\,\sqrt{\langle x^2\rangle\langle p_x^2\rangle}=\frac{Rp_0}{mc}, \tag{3.4}$$

therefore, $p_0=mc\,\epsilon/R$. From Eqs. (3.2), (3.3), and (3.4), the value of H_0 is given by

$$H_0=\frac{p_0^2}{4m\gamma}=\frac{mc^2}{4\gamma}\left(\frac{\epsilon}{R}\right)^2. \tag{3.5}$$

The space charge density of the beam is expressed via the distribution function after integration over particle momentum:

TABLE I. Ratio of space charge density of the beam at the axis to average value of density $k=\rho_0/\overline{\rho}$ for different particle distributions.

Particle distribution	k
$\rho(r)=\rho_0$	1
$\rho(r)=\rho_0\left[1-\frac{2}{3}\left(\frac{r}{R}\right)^2\right]$	$\frac{4}{3}$
$\rho(r)=\rho_0\left[1-\frac{1}{2}\left(\frac{r}{R}\right)^2\right]^2$	1.5
$\rho(r)=\rho_0\exp\left(-2\,\frac{r^2}{R^2}\right)$	2

$$\rho(x,y)=q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,p_x,p_y)dp_x\,dp_y$$
$$=\rho_0\,\exp\left(-q\,\frac{U_{\text{ext}}+\gamma^{-2}U_b}{H_0}\right),$$
$$\rho_0=2\pi mf_0H_0\gamma q. \tag{3.6}$$

In Eq. (3.6) ρ_0 is the value of the space charge particle density in the center of the beam. The value of ρ_0 is unknown at this point due to the unknown space charge potential of the beam U_b. Let us introduce an average value of space charge density of the beam:

$$\overline{\rho}=\frac{I}{\beta c\pi R^2}. \tag{3.7}$$

In general, the particle density at the axis ρ_0 differs from the average value of space charge density $\overline{\rho}$ as a factor of k:

$$\rho_0=k\overline{\rho}, \tag{3.8}$$

where parameter k has typical values presented in Table I. Taking into account the adopted relationship (3.8), the value of f_0 is expressed as follows:

$$f_0=k\,\frac{2I}{\pi^2\beta qm^2c^3\epsilon^2}. \tag{3.9}$$

IV. SELF-CONSISTENT SPACE CHARGE POTENTIAL OF THE BEAM

To find the self-consistent particle distribution, one has to solve Poisson's equation for an unknown space charge potential of the beam. Let us introduce dimensionless variables:

$$V_{\text{ext}}=\frac{qU_{\text{ext}}}{H_0},\quad V_b=\frac{qU_b}{H_0},\quad \xi=\frac{r}{a}, \tag{4.1}$$

where a is the radius of the channel. Poisson's equation in cylindrical polar coordinates is

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi}+\frac{\partial^2 V_b}{\partial \xi^2}+\frac{1}{\xi^2}\frac{\partial^2 V_b}{\partial \varphi^2}=-\Phi_0 \exp-(V_{\mathrm{ext}}+V_b\gamma^{-2}),$$

$$\Phi_0=16k\gamma\frac{I}{\beta I_c}\left(\frac{a}{\epsilon}\right)^2. \qquad (4.2)$$

The unknown potential V_b can be expressed as a Fourier-Bessel series,

$$V_b=V_0+\overline{V_b},$$

$$\overline{V_b}=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}J_n(\upsilon_{nm}\xi)(A_{nm}\cos n\varphi+B_{nm}\sin n\varphi), \qquad (4.3)$$

where $J_n(x)$ is a Bessel function and υ_{nm} is the mth root of the equation $J_n(x)=0$. Expansion (4.3) satisfies the Dirichlet boundary condition at the conductive surface of a round pipe $V_b(a)=V_0$. Constant V_0 is defined below such that the total potential of the structure vanishes at the axis:

$$V_{\mathrm{ext}}(0,\varphi)+\frac{\overline{V_b}(0,\varphi)}{\gamma^2}+\frac{V_0}{\gamma^2}=0. \qquad (4.4)$$

To find an approximate solution of Poisson's equation, let us take the first term in the near-axis expansion of exponential function:

$$\exp\left(-V_{\mathrm{ext}}-\frac{V_b}{\gamma^2}\right)\approx 1-V_{\mathrm{ext}}-\frac{V_b}{\gamma^2}. \qquad (4.5)$$

The left side of Poisson's equation is

$$\frac{1}{\xi}\frac{\partial V_b}{\partial \xi}+\frac{\partial^2 V_b}{\partial \xi^2}+\frac{1}{\xi^2}\frac{\partial^2 V_b}{\partial \varphi^2}=-\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\upsilon_{nm}^2 J_n(\upsilon_{nm}\xi)\times(A_{nm}\cos n\varphi+B_{nm}\sin n\varphi), \qquad (4.6)$$

therefore, Poisson's equation with exponential expansion (4.5) becomes

$$V_0+\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[1+\frac{\upsilon_{nm}^2\gamma^2}{\Phi_0}\right]J_n(\upsilon_{nm}\xi)(A_{nm}\cos n\varphi+B_{nm}\sin n\varphi)=\gamma^2(1-V_{\mathrm{ext}}). \qquad (4.7)$$

We introduce dimensionless value of beam brightness:

$$b=\frac{2}{\beta\gamma}\frac{I}{\epsilon^2}\frac{R^2}{I_c}. \qquad (4.8)$$

Parameter b is the figure of merit for a space charge dominated beam, obeying the Kapchinsky-Vladimirsky (KV) envelope equation in an ideal uniform focusing channel. Space charge dominated beam transport is performed if $b \gg 1$, otherwise the emittance dominated regime is fulfilled for $b \ll 1$. Therefore, in the case of high brightness beam transport, parameter Φ_0 in Poisson's equation (4.2) is much larger than unity:

$$\Phi_0=8k\gamma^2 b\left(\frac{a}{R}\right)^2 \gg 1. \qquad (4.9)$$

It is possible to simplify with an approximation to Poisson's equation (4.7). For monotonous space charge distributions, the values of coefficients A_{nm} and B_{nm} in space charge potential expansion (4.3) vanish quickly with increasing of indexes n and m. Roots of the Bessel function are slow functions of numbers n,m: $\upsilon_{01}=2.408$; $\upsilon_{11}=3.83$, $\upsilon_{21}=5.13$, $\upsilon_{02}=5.52$. The ratio of the beam radius to the aperture of the channel has a typical value of $R/a\approx 0.5$. Consequently, the following factor in Eq. (4.7) can be approximated as a constant, close to unity:

$$1+\frac{\upsilon_{nm}^2\gamma^2}{\Phi_0}=1+\frac{\upsilon_{nm}^2}{8bk}\left(\frac{R}{a}\right)^2\approx 1+\delta, \quad \delta=\frac{1}{bk}\ll 1. \qquad (4.10)$$

The above factor can be taken out of the sum in Eq. (4.7):

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left(1+\frac{\upsilon_{nm}^2\gamma^2}{\Phi_0}\right)J_n(\upsilon_{nm}\xi)(A_{nm}\cos n\varphi+B_{nm}\sin n\varphi)\approx(1+\delta)\overline{V_b}. \qquad (4.11)$$

With Eq. (4.11), the approximation to Poisson's equation is given by

$$V_0+(1+\delta)\overline{V_b}=\gamma^2(1-V_{\mathrm{ext}}). \qquad (4.12)$$

The external potential in general can be represented as an expansion on multipole components:

$$U_{\mathrm{ext}}=\sum_{n=2}^{\infty}\left(\frac{r}{a}\right)^n(\overline{U_n}\cos n\varphi+\bar{\bar{U}}_n\sin n\varphi). \qquad (4.13)$$

Expression (4.13) vanishes on axis, therefore the unknown constant in the space charge potential expression is given by Eqs. (4.4) and (4.12) as

$$V_0=-\overline{V_b}(0,\varphi)=-\frac{\gamma^2}{\delta}. \qquad (4.14)$$

Finally, the self-consistent space charge dominated beam potential near axis is

$$V_b=\overline{V_b}+V_0=-\frac{\gamma^2}{1+\delta}V_{\mathrm{ext}}. \qquad (4.15)$$

The same relationship is valid for the electric field $\vec{E}=-\mathrm{grad}\, U$:

$$\vec{E}_b=-\frac{\gamma^2}{1+\delta}\vec{E}_{\mathrm{ext}}. \qquad (4.16)$$

From Eq. (4.15) it follows that the space charge dominated beam always compensates for the focusing field in the beam core regardless of the applied external focusing potential. This fact is well known for channels with linear focusing field [3], but now it is shown also for an arbitrary focusing field. In the above derivations, there were no assumptions about the specific features of the focusing field. The particle distribution of the bright matched beam always tends to such a shape that the space charge beam potential is opposite to the external focusing potential. This phenomenon is known from plasma physics as Debye shielding for nonneutral plasmas. For high brightness beams, the Debye length is much smaller than the beam radius [4] and hence, as demonstrated by Eq. (4.15), space charge dominated beam always compensate for the focusing field. Therefore our analysis gives us the possibility of matching a bright nonuniform particle beam with the channel through the selection of multipole focusing field components. In Sec. VI, this approach will be used to provide halo-free beam transport of a nonuniform bright beam.

The second approximation to the self-consistent potential V_b can be obtained by taking one more term in the expansion of the exponential function:

$$\exp(-V_{\text{ext}}-\gamma^{-2}V_b)\approx 1-V_{\text{ext}}-\gamma^{-2}V_b+\frac{(V_{\text{ext}}+\gamma^{-2}V_b)^2}{2}. \tag{4.17}$$

Repeating similar derivations leading to Eq. (4.12), the second approximation to the space charge potential is defined by

$$V_0+(1+\delta)\overline{V_b}=\gamma^2(1-V_{\text{ext}})+\frac{\gamma^2}{2}\left(V_{\text{ext}}+\frac{V_0}{\gamma^2}+\frac{\overline{V_b}}{\gamma^2}\right)^2. \tag{4.18}$$

Substituting the axial condition $V_{\text{ext}}(0,\varphi)=0$, $\overline{V_b}(0,\varphi)=-V_0$ into Eq. (4.18), the value of the constant $V_0=-\gamma^2/\delta$ appears to be the same as in the first approximation. The quadratic equation for unknown space charge potential is

$$V_b^2+V_b 2\gamma^2(V_{\text{ext}}-1-\delta)+\gamma^4 V_{\text{ext}}(V_{\text{ext}}-2)=0. \tag{4.19}$$

The solution of Eq. (4.19) is a second approximation to the space charge potential:

$$V_b=\gamma^2(1+\delta-V_{\text{ext}})-\gamma^2\sqrt{(1+\delta-V_{\text{ext}})^2-V_{\text{ext}}(V_{\text{ext}}-2)}. \tag{4.20}$$

For small values of $V_{\text{ext}}\ll 1$, expression (4.20) transforms to a linear relationship between space charge potential and external focusing potential (4.15). In the limit of a very high brightness beam $\delta\to 0$, the second approximation (4.20) gives the same result, $V_b=-\gamma^2 V_{\text{ext}}$, as the linear approximation (4.15). This means that the linear approximation (4.15) becomes more valid with increasing beam brightness.

In general, Poisson's equation for $\delta\ll 1$ can be written as follows:

$$\frac{\delta}{\gamma^2}\overline{V_b}=\exp\left(-V_{\text{ext}}-\frac{\overline{V_b}}{\gamma^2}-\frac{\overline{V_0}}{\gamma^2}\right). \tag{4.21}$$

The expression under the exponential function vanishes on axis, therefore, the unknown constant V_0 in the adopted model always equals the value of (4.14) due to the resulting equation

$$-\frac{\delta}{\gamma^2}V_0=1. \tag{4.22}$$

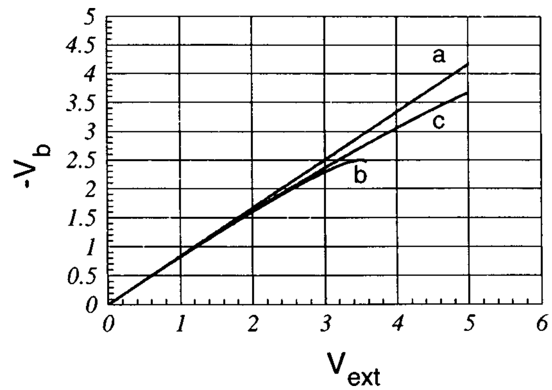

FIG. 1. Self-consistent potential of a high brightness beam V_b as a function of applied focusing potential V_{ext} for $\delta=0.2$, $\gamma=1$: (a) linear approximation [Eq. (4.15)], (b) second order approximation [Eq. (4.20)], (c) numerical solution of Eq. (4.23).

Using the value of constant V_0, Poisson's equation for a high brightness beam is as follows:

$$1+\frac{\delta}{\gamma^2}V_b=\exp\left(-V_{\text{ext}}-\frac{V_b}{\gamma^2}\right). \tag{4.23}$$

Higher order approximations to the space charge potential V_b can be obtained from Eq. (4.23) by holding more terms in the expansion of the exponential function or via numerical solution of Eq. (4.23). In the extreme case of a very high brightness beam, Eq. (4.23) gives the same result as linear (4.15) and second order (4.20) equations:

$$1=\exp\left(-V_{\text{ext}}-\frac{V_b}{\gamma^2}\right),\quad V_b=-\gamma^2 V_{\text{ext}}. \tag{4.24}$$

In this case the space charge potential of the beam completely compensates for focusing field.

In Fig. 1 results of different approximations to the self-consistent space charge potential of the beam for the value of $\delta=0.2$ are presented. Both first and second approximations are close to the exact numerical solution of Eq. (4.23) up to $V_{\text{ext}}<3$. The second order approximation is valid until the determinant in Eq. (4.20) is positive:

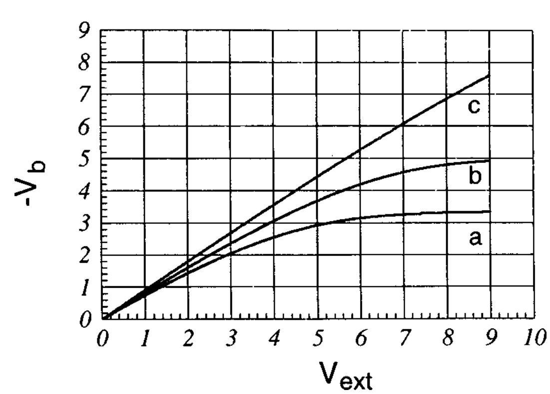

FIG. 2. Results of the numerical solution of Eq. (4.23) for a self-consistent potential of a high brightness beam: (a) $\delta=0.3$; (b) $\delta=0.2$; (c) $\delta=0.1$.

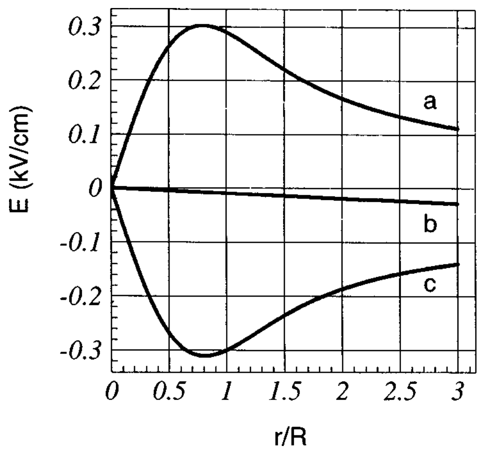

FIG. 3. (a) Space charge field $\gamma^{-2}E_b$ [Eq. (5.4)], (b) total field $E_{\rm tot}$ [Eq. (5.3)], and (c) required focusing field $E_{\rm ext}$ [Eq. (5.5)] for 150 keV, 100 mA, 0.1π cm mrad proton beam with a Gaussian distribution function.

$$V_{\rm ext} \leqslant \frac{(1+\delta)^2}{2\delta}. \tag{4.25}$$

In Fig. 2 results of numerical solution of Poisson's equation (4.23) for different values of beam brightness are presented. As seen, with increasing beam brightness, an exact numerical solution of Poisson's equation becomes close to the linear relationship between space charge potential and external potential.

The space charge distribution of a matched beam can be derived from Poisson's equation via a known space charge potential of the beam

$$\rho_b = -\epsilon_0 \Delta U_b = \frac{\epsilon_0}{1+\delta}\,\gamma^2 \Delta U_{\rm ext}. \tag{4.26}$$

The space charge density of high brightness beam is defined by the external focusing potential function $U_{\rm ext}$ and is a weak function of the phase space density of the beam. It provides an easy way to find a self-consistent particle distribution in the channel with a given focusing potential.

V. COMPARISON WITH KNOWN SOLUTIONS

Let us check the developed approach with Gaussian beam matched with nonlinear channel [1]. Consider an inverse problem and define the required external field to maintain a high brightness beam with a Gaussian particle distribution:

$$f = f_0 \exp\left(-2\,\frac{x^2+y^2}{R^2} - 2\,\frac{p_x^2+p_y^2}{p_0^2}\right). \tag{5.1}$$

According to the general approach suggested in [1], the solution of Vlasov's equation for a Gaussian distribution function (5.1) provides an expression for the total field of the structure:

$$U(x,y) = \frac{1}{\gamma}\,\frac{mc^2}{q}\,\frac{\epsilon^2}{R^4}\left(\frac{x^2+y^2}{2}\right), \tag{5.2}$$

$$E_{\rm tot} = -\frac{\partial U}{\partial r} = -\frac{1}{\gamma}\,\frac{mc^2}{q}\,\frac{\epsilon^2}{R^4}\,r. \tag{5.3}$$

The space charge field of the Gaussian beam is attained from Poisson's equation:

$$E_b = -\frac{\partial U_b}{\partial r} = \frac{I}{2\pi\epsilon_0 \beta c}\,\frac{1}{r}\left[1-\exp\left(-2\,\frac{r^2}{R^2}\right)\right]. \tag{5.4}$$

The external focusing field required to maintain a beam with a Gaussian distribution is

$$E_{\rm ext} = E_{\rm tot} - \frac{E_b}{\gamma^2} = -\frac{mc^2}{qR\gamma}\left\{\frac{\epsilon^2 r}{R^3} + 2\,\frac{I}{I_c \beta\gamma}\,\frac{R}{r}\left[1-\exp\left(-2\,\frac{r^2}{R^2}\right)\right]\right\}. \tag{5.5}$$

Figure 3 illustrates the relationships between the space charge field of the beam, Eq. (5.4), the total field, Eq. (5.3), and the focusing field of the structure, Eq. (5.5), for a Gaussian beam with parameter $b=35$. As can be seen, the external field and the space charge field of the beam are close to each other with opposite sign, as described by Eq. (4.16).

Now let us apply the results of Sec. IV and define a self-consistent particle distribution via a known focusing field (5.5). Application of formula (4.26) gives

$$\rho_b = -\frac{\epsilon_0}{(1+\delta)}\,\gamma^2\,\frac{1}{r}\,\frac{\partial}{\partial r}\,(rE_{\rm ext}) = \frac{2I}{\beta c \pi R^2}\,\exp\left(-2\,\frac{r^2}{R^2}\right)\left[\frac{1}{(1+\delta)}\left(1+\frac{I_c \epsilon^2 \beta\gamma}{4IR^2\,\exp[-2(r^2/R^2)]}\right)\right]. \tag{5.6}$$

The expression in square brackets in Eq. (5.6) is always close to unity near the axis $r \ll R$. But for small values of $\delta < 10^{-2}$, the expression in square brackets is close to unity far away from the axis until $r \leqslant R$:

$$\left[\frac{1}{(1+\delta)}\left(1+\frac{I_c \epsilon^2 \beta\gamma}{4IR^2 \exp[-2(r^2/R^2)]}\right)\right] = \frac{1+\delta/\exp[-2(r^2/R^2)]}{1+\delta} \underset{\substack{\delta<0.01\\ r\leqslant R}}{\longrightarrow} 1. \tag{5.7}$$

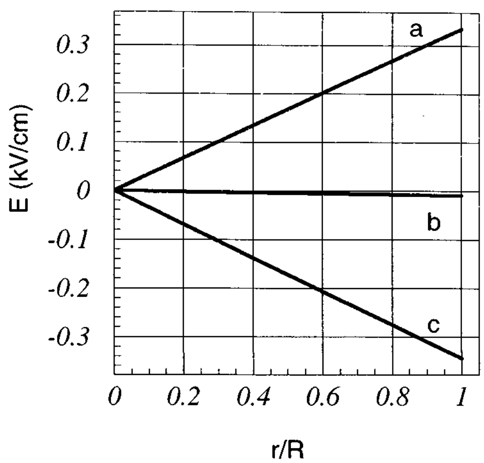

FIG. 4. (a) Space charge field, (b) total field and (c) required focusing field for 150 keV, 100 mA, 0.1π cm mrad proton beam with KV distribution function.

The rest of the expression (5.6) describes the beam with a Gaussian distribution

$$\rho_b = \frac{2I}{\beta c \pi R^2} \exp\left(-2\frac{r^2}{R^2}\right), \tag{5.8}$$

which coincides with initial suggestion (5.1).

The performed example demonstrates the validity of the developed approach to determine a self-consistent particle distribution. Formula (4.26) gives the correct expression for the space charge distribution of a matched beam with a high value of beam brightness, $b > 50$, within the beam size $r \leqslant R$.

Similar results are found for a beam with KV distribution. Figure 4 illustrates the relationships between the space charge field of the beam, the total field, and the focusing field of the structure for a KV beam with parameter $b = 35$. Let us note that Eqs. (4.15), (4.16), and (4.26) are always valid for the KV beam.

VI. BEAM TRANSPORT IN A FOCUSING CHANNEL WITHOUT HALO FORMATION

The above analysis results in a solution to an important problem: providing conditions for halo-free nonuniform beam transport in an alternating-gradient focusing channel. Nonlinear space charge forces of a high intensity beam produce strong emittance growth and halo formation in a linear focusing channel due to mismatch of the beam profile with the focusing field (see Fig. 5). In Ref. [2] it was shown that incorporation of a duodecapole component in a pure quadrupole channel results in suppression of emittance growth. A special case is a four-vane quadrupole structure, where the shape of the electrodes is modified to create a multipole field distribution as shown in Fig. 6 [5].

Let us consider a uniform four vanes structure with potential

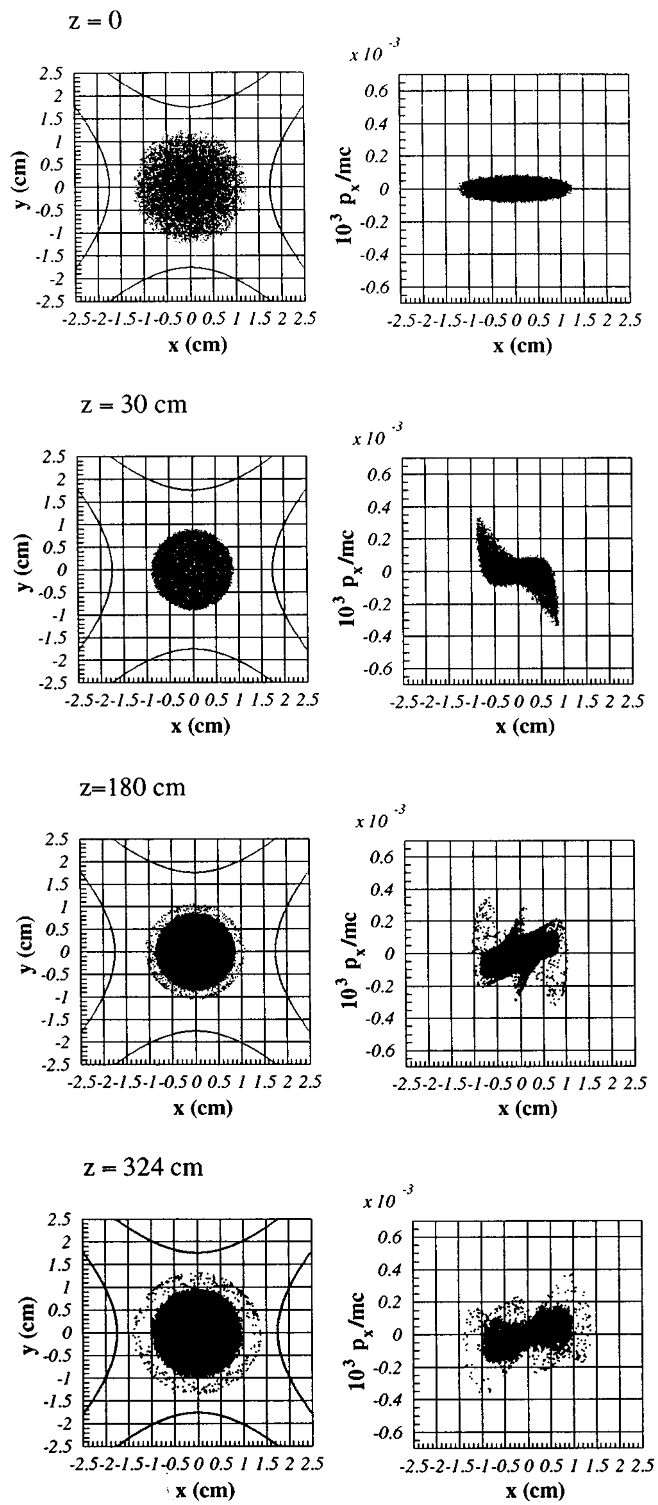

FIG. 5. Emittance growth and halo formation of the 150 keV, 100 mA, 0.06π cm mrad proton beam with parabolic distribution function (6.15) in a four vanes quadrupole structure with a field gradient of $G_2 = 50$ kV/cm^2.

$$U(r,\varphi,t) = \left(\frac{G_2}{2} r^2 \cos 2\varphi + \frac{G_6}{6} r^6 \cos 6\varphi\right) \sin \omega_0 t, \tag{6.1}$$

where G_2 is a quadrupole gradient, G_6 is a duodecapole

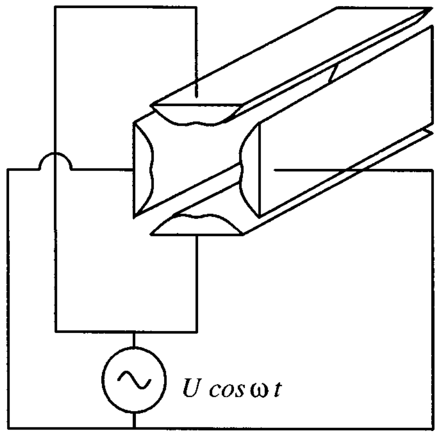

FIG. 6. Proposed four vane quadrupole structure with a duodecapole field component [5].

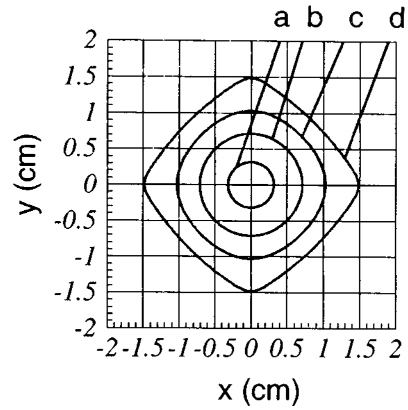

FIG. 7. Lines of equal values of the function $C=\frac{1}{2}r^2+\zeta r^6\cos 4\varphi+(\zeta^2/2)r^{10}$ for $\zeta=-0.03$: (a) $C=0.05$, (b) $C=0.25$, (c) $C=0.5$, and (d) $C=0.85$.

component, and $\omega_0=2\pi c/\lambda$ is an operational frequency. The electrical field of the structure is given by

$$\vec{E}(r,\varphi,t)=[-\vec{i}_r(G_2 r\cos 2\varphi+G_6 r^5\cos 6\varphi)+\vec{i}_\varphi(G_2 r\sin 2\varphi+G_6 r^5\sin 6\varphi)]\sin\omega_0 t. \tag{6.2}$$

Particle trajectories in the field (6.2) can be represented as a combination of a slow variation of particle position with fast oscillations of small amplitude. If phase advance of the particle oscillation per period of field variation is much smaller than 2π, the oscillating field (6.1) can be replaced by an effective scalar potential of the structure [6]

$$U_{\rm ext}(\vec{r})=\frac{q}{4m\gamma}\frac{E_0^2(\vec{r})}{\omega_0^2}, \tag{6.3}$$

which describes the averaged motion of particle. For the considered structure, the effective potential is

$$U_{\rm ext}(r,\varphi)=\frac{mc^2}{q}\frac{\mu_0^2}{\lambda^2}\left[\frac{1}{2}r^2+\zeta r^6\cos 4\varphi+\frac{\zeta^2}{2}r^{10}\right], \tag{6.4}$$

where μ_0 is a smooth transverse oscillation frequency and ζ is a ratio of field components:

$$\mu_0=\frac{qG_2\lambda^2}{\sqrt{8}\pi mc^2\sqrt{\gamma}},\quad \zeta=\frac{G_6}{G_2}. \tag{6.5}$$

The effective potential (6.4) is axially nonsymmetric and a highly nonlinear function of radius. Equipotential lines $U_{\rm ext}(r,\varphi)=C$ are circles near the axis and are transformed to a 45° skewed square far from the axis (see Fig. 7).

Application of Eq. (4.26) gives an expression for the self-consistent space charge distribution of the beam in the structure:

$$\rho_b=\rho_0(1+10\zeta r^4\cos 4\varphi+25\zeta^2 r^8), \tag{6.6}$$

$$\rho_0=\frac{2\gamma^2}{(1+\delta)}\frac{mc^2}{q}\frac{\epsilon_0\mu_0^2}{\lambda^2}. \tag{6.7}$$

Integrating the space charge density over radius and azimuth angle $0\leqslant r\leqslant R$, $0\leqslant\varphi\leqslant 2\pi$ gives the total number of transported particles per unit length:

$$N=\frac{I}{\beta c}=\frac{\rho_0}{q}\int_0^R\int_0^{2\pi}(1+10\zeta r^4\cos 4\varphi+25\zeta^2 r^8)r\,dr\,d\varphi=\frac{\pi\rho_0}{q}(R^2+5\zeta^2R^{10}). \tag{6.8}$$

From Eq. (6.8), the space charge particle density at the beam center and parameter k are as follows:

$$\rho_0=\frac{1}{(1+5\zeta^2R^8)}\frac{I}{\beta c\pi R^2}, \tag{6.9}$$

$$k=\frac{1}{(1+5\zeta^2R^8)}. \tag{6.10}$$

Comparison of Eq. (6.9) with Eq. (6.7) gives the required value of the focusing gradient to provide beam confinement:

$$G_2=\sqrt{8}\pi\frac{mc^2}{qR\lambda}\left(\frac{\epsilon^2}{R^2}+\frac{2I}{I_c\beta\gamma(1+5\zeta^2R^8)}\right)^{1/2}. \tag{6.11}$$

In Fig. 8 an example of particle distribution (6.6) with the ratio of field components $\zeta=-0.03$ is presented. To gener-

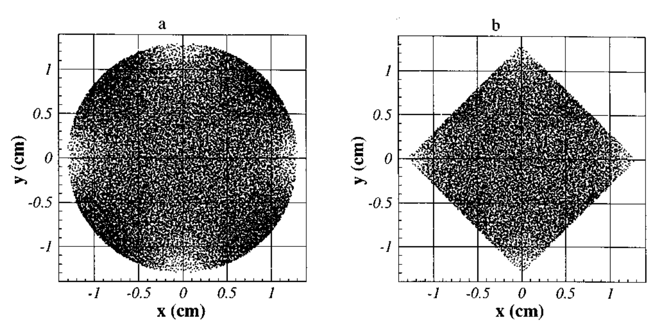

FIG. 8. Self-consistent particle distribution $\rho_b=\rho_0(1+10\zeta r^4\cos 4\varphi+25\zeta^2r^8)$ of the matched beam in a quadrupole channel with a duodecapole component with parameter $\zeta=-0.03$: (a) without truncation, (b) with truncation.

ate the particle distribution, the area $-R_{\max}<x<R_{\max}$, $-R_{\max}<y<R_{\max}$, where $R_{\max}=1.3$ cm, was covered by a grid with small steps $\Delta x=0.01R_{\max}$, $\Delta y=0.01R_{\max}$. Inside every elementary mesh of the grid, the fraction of modeling particles $\Delta N(x,y)/N_0$ was defined:

$$\frac{\Delta N(x,y)}{N_0}=[1+10\zeta(x^4-6x^2y^2+y^4)+25\zeta^2(x^2+y^2)^4]\,\frac{\Delta x\ \Delta y}{4R_{\max}^2}, \qquad (6.12)$$

where the relationship $r^4\cos 4\varphi=x^4-6x^2y^2+y^4$ was used. Parameter N_0 in Eq. (6.12) defines the density of modeling particles through the number of particles at beam center $\Delta N(0,0)$:

$$N_0=\Delta N(0,0)\,\frac{4R_{\max}^2}{\Delta x\ \Delta y}. \qquad (6.13)$$

The defined number of particles in every elementary mesh $\Delta N(x,y)$ were uniformly distributed inside the mesh area.

Self-consistent particle distribution (6.6) has a fourfold symmetry [see Fig. 8(a)]. Every 45° variation of azimuth angle φ results in a change of the particle distribution from a decreasing to an increasing function of radius and vice versa. Equipotential lines of the self-potential of a high brightness beam are close to that of an external focusing potential, which are shown in Fig. 7. To treat the matched beam, it is necessary to bound the beam along equipotential lines. In this case, the space charge forces at the beam boundaries will be kept close to that of an unbounded beam, remaining perpendicular to the equipotential lines. Therefore, the beam boundaries have to be 45° skewed square [see Fig. 8(b)], as suggested in Ref. [2].

In Fig. 9, the results of a particle-in-cell simulation of matched beam transport in a quadrupole channel with a duodecapole field component are presented. The beam distribution function was a product of the matched beam profile in real space (6.6) and Gaussian distribution function in momentum:

$$f=f_0(1+10\zeta r^4\cos 4\varphi+25\zeta^2r^8)\exp\left(-2\,\frac{R^2}{\epsilon^2}\,\frac{p_x^2+p_y^2}{m^2c^2}\right). \qquad (6.14)$$

The beam was truncated along a 45° skewed square in real space, as shown in Fig. 8(b). The value of the dimensionless beam brightness was chosen to be $b=10^2$. Particle trajectories were integrated in the field, which was a combination of the time-dependent potential (6.1) and the space charge potential of the beam utilizing the leap-frog method. The space charge potential of the beam was calculated at every integration step employing a double fast Fourier transformation [1]. The value of the field gradient $G_2=48\ \mathrm{kV/cm^2}$ was defined by Eq. (6.11) and the value of the duodecapole component was $G_6=-1.3\ \mathrm{kV/cm^6}$, which corresponds to parameter $\zeta=-0.03$. As shown in Fig. 9, the distribution is conserved, which proves that it is matched with the focusing channel.

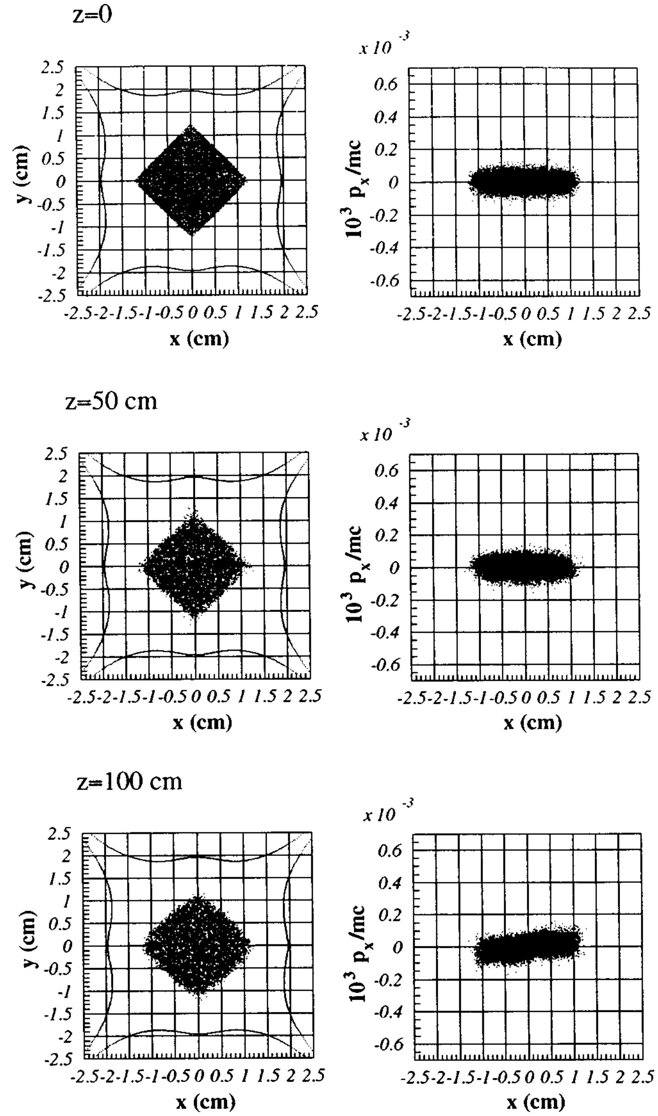

FIG. 9. Emittance conservation of the 150 keV, 100 mA, 0.06π cm mrad proton beam with a matched distribution function (6.14) in a four vane quadrupole structure with field gradient $G_2=48\ \mathrm{kV/cm^2}$ and duodecapole component $G_6=-1.3\ \mathrm{kV/cm^6}$.

The realistic beam distribution is a monotonically decreasing function of radius, which differs from distribution (6.6). A good approximation to the realistic beam is a parabolic distribution in phase space [1,2]:

$$f=f_0\left(1-\frac{x^2+y^2}{2R^2}-\frac{p_x^2+p_y^2}{2p_0^2}\right). \qquad (6.15)$$

The parabolic distribution (6.15) has a projection in configuration space close to a truncated Gaussian distribution:

$$\rho_b=\frac{3I}{2\pi c\beta R^2}\left(1-\frac{r^2}{2R^2}\right)^2. \qquad (6.16)$$

Required values of quadrupole gradient and duodecapole components to provide matching of such a beam with the channel are found analogously to Ref. [2]:

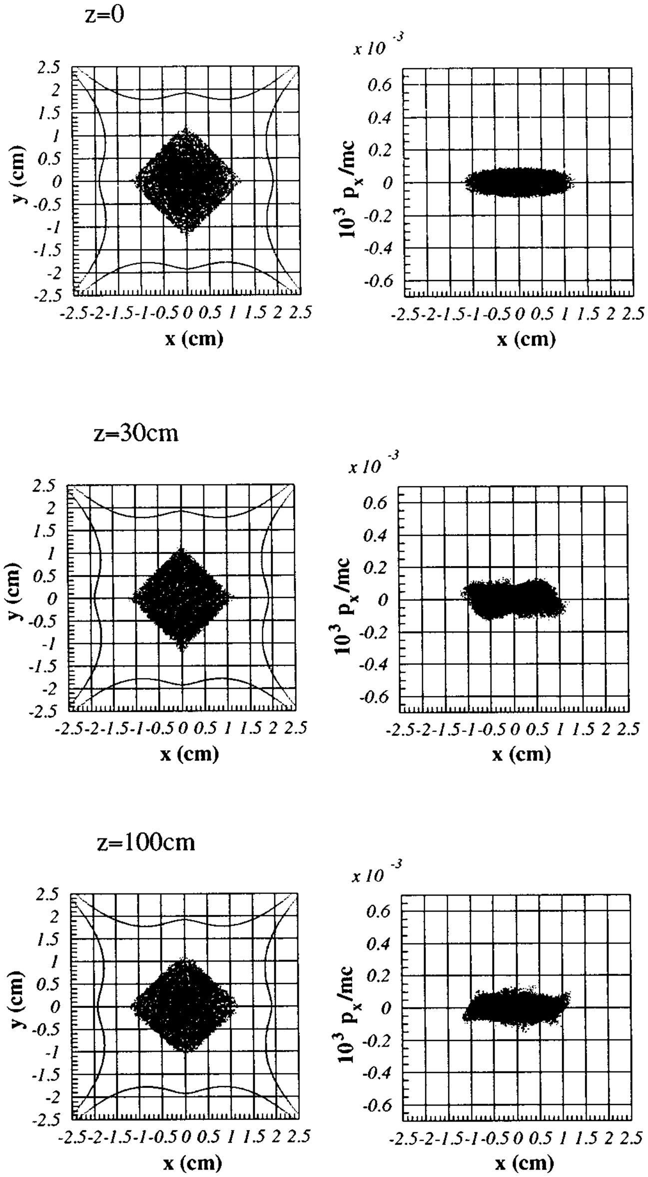

FIG. 10. Emittance conservation of the 150 keV, 100 mA, 0.06π cm mrad proton beam with a truncated parabolic distribution function (6.15) in a four vane quadrupole structure with field gradient $G_2 = 50$ kV/cm^2 and duodecapole component $G_6 = -1.9$ kV/cm^6.

$$G_2 = \frac{\sqrt{8}\pi mc^2}{q\lambda R}\left(\frac{\epsilon^2}{R^2} + \frac{3I}{I_c\beta\gamma}\right)^{1/2}, \tag{6.17}$$

$$G_6 = -\frac{G_2}{12\beta\gamma R^4}\frac{I}{I_c}\left(\frac{\epsilon^2}{R^2} + \frac{3I}{I_c\beta\gamma}\right)^{-1}. \tag{6.18}$$

In Fig. 10, the results of the beam transport simulation with a parabolic distribution in a quadrupole channel with field components $G_2 = 52$ kV/cm^2, $G_6 = -1.9$ kV/cm^6 are presented. To make the beam distribution as close to a matched beam distribution as possible, the beam boundaries were truncated along equipotential lines in the same manner as was done for the distribution (6.6). Space charge field of the beam

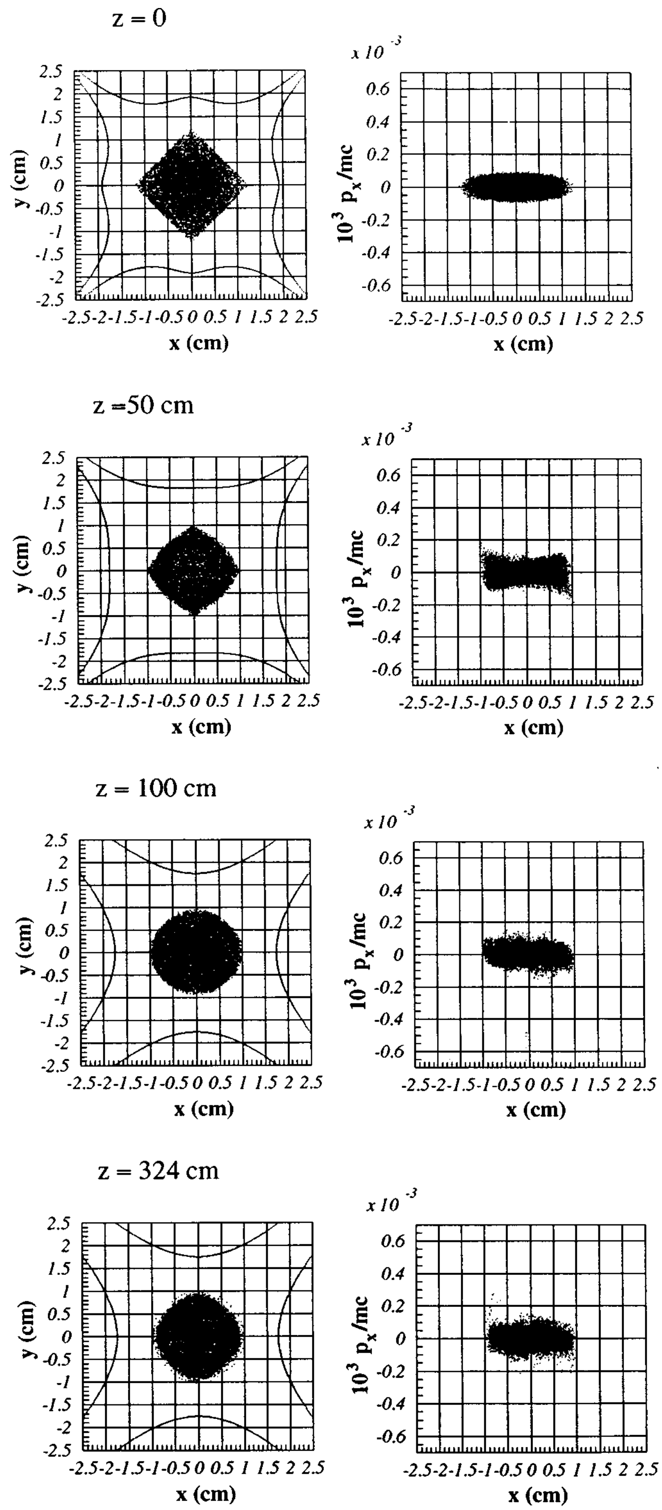

FIG. 11. Adiabatic matching to avoid halo formation of 150 keV, 100 mA, 0.06π cm mrad proton beam in a four vane quadrupole structure with field gradient $G_2 = 50$ kV/cm^2 and adiabatic decline of the duodecapole component from $G_6 = -1.9$ kV/cm^6 to zero for the distance $L = 100$ cm.

$$U_b = -\frac{3}{2}\frac{mc^2}{q}\frac{I}{I_c\beta}\left(\frac{r^2}{R^2} - \frac{r^4}{4R^4} + \frac{r^6}{36R^6}\right), \tag{6.19}$$

includes term $\sim r^4$, which is not present in an effective potential (6.4). Therefore, the beam with a parabolic distribu-

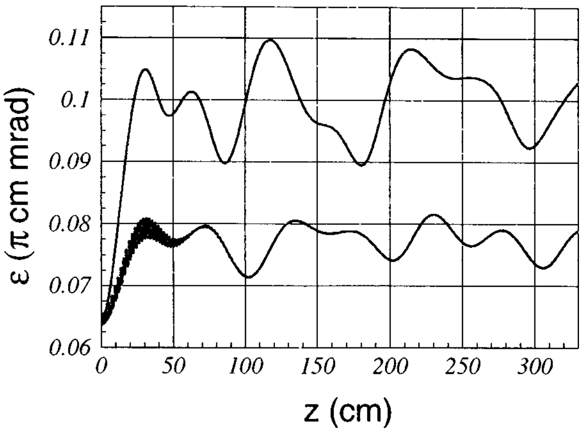

FIG. 12. Beam emittance growth in a four vanes structure with a pure quadrupole field (up) and in a quadrupole field with an adiabatic decline of the duodecapole component (bottom).

tion function cannot be exactly matched with the channel, which expresses itself as a small emittance distortion in phase space. Nevertheless, such a beam is much better matched with the channel than a round beam with the same distribution function in a pure quadrupole channel (compare with Fig. 5).

An extra possibility to employ a nonlinear focusing channel is connected with the adiabatic transformation of channel parameters [2]. In Fig. 11, the results of the beam dynamics simulation in an adiabatic nonlinear matcher are presented. The value of the quadrupole gradient $G_2 = 52$ kV/cm^2 was kept constant along the channel. The duodecapole component G_6 was adiabatically changed from the value -1.9 kV/cm^6, as required by matched conditions, to zero for the distance $L = 100$ cm. After the nonlinear matching section $z > L$, the channel was a pure quadrupole and the beam was transported 224 cm more to check the results of the transformation. As can be seen, the beam profile in real space is modified from a square to a circular shape and follows the adiabatic change of the effective potential. rms beam emittance growth in a nonlinear transformer is 15%, which is substantially smaller than the 50% emittance growth in a pure quadrupole channel (see Fig. 12). The final beam emittance and beam profile are matched without serious phase space portrait distortion and halo formation. After transformation, the beam can be transported in a conventional structure with a linear focusing field.

VII. CONCLUSIONS

The self-consistent space charge potential of a high brightness beam is derived in the case of an arbitrary potential for a continuous focusing channel. It is shown that a matched beam always tends to compensate for the applied potential. This is a manifestation of Debye shielding for nonneutral plasmas. A simple formula is given that demonstrates the shielding effect of an arbitrary focusing potential by a self-consistent beam field. A four-vane quadrupole structure with a multipole component of the 6th order (duodecapole component) is analyzed to prevent space charge dominated beam emittance growth. In such a structure, the matched beam profile has to be close to square instead of the conventional circle beam cross section. Adiabatic change of a nonlinear focusing field along the beam structure results in gradual transformation of an initially nonuniform beam distribution into a distribution matched with the linear focusing channel. The given analysis provides matched conditions for nonuniform high brightness beam transport without serious emittance growth and halo formation.

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